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**STRATEGY OF FINDING THE MAXIMUM AND MINIMUM VALUES OF THE
FUNCTION OF N- VARIABLES WITH AND WITHOUT CONSTRAINT**

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ABSTRACT

This paper is about the finding the maximum and minimum values of the function of n- variables with and without constraint.

KEYWORDS: stationary values, Lagrange multiplier , Euler – Lagrange equations.

INTRODUCTION

A lot of real world problems involve finding a minimum or a maximum of a given function , the maximum and minimum values of a function , which also known the extreme are defined to be the largest and smallest value that the function takes at a given points, see ([1],[2]) . The maximum and minimum problems with or without restriction or constraint have great importance and many applications in different areas of applied science and also in real life , for example they used to find the shortest path from point A to B (minimizing distance in navigation) , to invest money for biggest return (maximizing profit in economics) also they have great applications in geography as they used to determine the tallest peak in a mountain region.

Definition1: A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$, $\forall x \in D$, where D is the domain of f . The number $f(c)$ is called the maximum value of f on D . Similarly, f has an absolute minimum (or global minimum) at c if $f(c) \leq f(x)$, $\forall x \in D$ and the number $f(c)$ is called the minimum value of f on D .

Definition2: A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c . Similarly, f has a local minimum (or relative minimum) at c if $f(c) \leq f(x)$ when x is near c .

Theorem 3: (The Extreme Value Theorem): If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem 4: (Fermat's Theorem): If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Definition5: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition6: A saddle point is a point in the range of a function that is a stationary point but not a local extremum.

Definition7: A point $(a; b)$ which is a maximum, minimum or saddle point is called a stationary point and the actual value at a stationary point is called the stationary value.

MAXIMIZATION AND MINIMIZATION OF FUNCTION OF N - VARIABLES WITH NO CONSTRAINT

Firstly we will discuss the method of finding the maximum and minimum values of the function of n - variables without constraint.

Consider the function $f(\underline{x})$, $\underline{x} = x_1, x_2, x_3, \dots, x_n$, the value of the function $f(\underline{x})$ when $x_1 = a_1, x_2 = a_2, x_3 = a_3, \dots, x_n = a_n$ is said to be stationary value of the function $f(\underline{x})$ if $\frac{\partial f}{\partial x_i} = 0, i = 1, 2, 3, \dots, n$.

When we expand the function $f(\underline{x})$, $\underline{x} = x_1, x_2, x_3, \dots, x_n$ at the point $(\underline{a} + \underline{h})$, by using Taylor's theorem where $h \ll 1$ we get

$$f(\underline{a} + \underline{h}) = f(\underline{a}) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \Big|_{h=a} h_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j} + O(h_i, h_j, h_k) \quad (1)$$

Let $f(\underline{a} + \underline{h}) - f(\underline{a}) = F$, and if $f(a_1, a_2, a_3, \dots, a_n)$ is given stationary values of $f(\underline{x})$, then $\sum_{i=1}^n \frac{\partial f}{\partial x_i} \Big|_{x=a} h_i = 0$ and then equation (1) will take the form:

$$\begin{aligned} F &= \frac{1}{2!} \left(\sum_{i=1}^n \sum_{j=1}^m h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j} \right) + \dots + O(h_i, h_j, h_k) \\ &= \frac{h F h^T}{2} + \dots + O(h_i, h_j, h_k) \end{aligned} \quad (2)$$

Where $\frac{h F h^T}{2} = \sum_{i=1}^n \sum_{j=1}^m h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}$ called the Quadratic Form in the variables $h_1, h_2, h_3, \dots, h_n$. For $(x_1, x_2, x_3, \dots, x_n)$ sufficiently close to

$(a_1, a_2, a_3, \dots, a_n)$ we can ignore the remainder terms, then $F = \frac{h F h^T}{2!}$, for which the value $h F h^T$

is non-zero, then F and $h F h^T$ have the same sign and then we have the following:

- If $h F h^T$ is positive for all $h_1, h_2, h_3, \dots, h_n$ where $h_1, h_2, h_3, \dots, h_n$ are not all zero, then $h F h^T$ is said to be positive definite and the corresponding stationary values are minimum.
- If $h F h^T$ is negative for all $h_1, h_2, h_3, \dots, h_n$ where $h_1, h_2, h_3, \dots, h_n$ are not all zero, then $h F h^T$ is said to be negative definite and the corresponding stationary values are maximum.

The necessary and sufficient conditions for $h F h^T$ to be positive definite is that :

$$f_{11} > 0, \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} > 0, \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} > 0, \dots, \begin{bmatrix} f_{11} & f_{12} & f_{13} & \dots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \dots & f_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \dots & f_{nn} \end{bmatrix} > 0 \quad (3)$$

Which it is the condition for the stationary values to be minimum.

And the necessary and sufficient conditions for $h F h^T$ to be negative definite is that :

$$f_{11} < 0, \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} > 0, \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} < 0, \dots \quad (4)$$

Which it is the condition for the stationary values to be maximum.

CONSTRAINED MAXIMUM AND MINIMUM VALUES

Now we want to discuss the method of finding the maximum and minimum values of the function of n - variables subject to given constraints.

There are many methods to solve such problems, but if the constrained problem has only equality constraints, then the method of solution is the method of Lagrange multipliers.

Definition8: Let $f(x)$ and $g(x)$ be functions of n - variables. An extreme value of $f(x)$ subject to the condition $g(x) = 0$, is called a constrained extreme value and $g(x) = 0$ is called the constraint.

Definition9: If $f: R^n \rightarrow R$ is a function of n variables, the Lagrangian function of f subject to the constraint $\phi(x_1, x_2, x_3, \dots, x_n) = 0$, is the function of $n+1$ variables of the form:

$f(x_1, x_2, x_3, \dots, x_n, \lambda) = \lambda \phi(x_1, x_2, x_3, \dots, x_n)$ Where λ is known as the Lagrange multiplier.

Consider the function $f(x)$, $x = x_1, x_2, x_3, x_n$ (5)

Which subject to

$$\phi_r(x_1, x_2, x_3, \dots, x_m) = 0, \quad 1 \leq r \leq m \quad (6)$$

Where $m < n$, then the necessary and sufficient conditions for a stationary values of $f(x)$, $x = x_{m+1}, x_{m+2}, x_{m+3}, \dots, x_n$ are:

$$\frac{\partial f}{\partial x_{m+1}} = \frac{\partial f}{\partial x_{m+2}} = \frac{\partial f}{\partial x_{m+3}} = \dots = \frac{\partial f}{\partial x_n} = 0 \quad (7)$$

Since $x_{m+1} = x_{m+1}(x_1, x_2, x_3, \dots, x_n)$ then the following equation hold

$$\frac{\partial f}{\partial x_s} + \sum_{t=1}^m \frac{\partial f}{\partial x_t} \frac{\partial x_t}{\partial x_s} = 0, \quad s = m+1, \dots, n, \quad x_i = x_i(x_{m+1}, x_{m+2}, \dots, x_n) \quad (8)$$

And the constraint can be written as :

$$\varphi_r (x_1(x_{m+1}, x_{m+2}, \dots, x_n), x_2(x_{m+1}, x_{m+2}, \dots, x_n), x_3, \dots, x_m(x_{m+1}, x_{m+2}, \dots, x_n)) = 0$$

Since $r = 1, \dots, m$, then the differentiation of φ_r with respect to $x_{m+1}, x_{m+2}, \dots, x_n$ gives

$$\frac{\partial \phi_r}{\partial x_s} + \sum_{t=1}^m \frac{\partial \phi}{\partial x_t} \frac{\partial x_t}{\partial x_s} = 0, \quad s = m+1, \dots, n \quad (9)$$

The values $x_1, x_2, x_3, \dots, x_n$ at stationary values satisfy the equations (8) and (9), if

$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ be any functions of $x_1, x_2, x_3, \dots, x_n$, then these values satisfy the equations

$$\frac{\partial f}{\partial x_t} + \sum_{i=1}^m \lambda_r \frac{\partial \phi_r}{\partial x_s} + \sum_{i=1}^m \left\{ \frac{\partial f}{\partial x_t} + \sum_{i=1}^m \lambda_r \frac{\partial \phi_r}{\partial x_s} \right\} \frac{\partial x_t}{\partial x_s} = 0, \quad s = m+1, \dots, n \quad (10)$$

Let $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ be defined by:

$$\frac{\partial f}{\partial x_t} + \sum_{r=1}^m \lambda_r \frac{\partial \phi_r}{\partial x_t} = 0, \quad t = 1, \dots, m \quad (11)$$

$$\frac{\partial f}{\partial x_s} + \sum_{r=1}^m \lambda_r \frac{\partial \phi_r}{\partial x_s} = 0, \quad s = m+1, \dots, n \quad (12)$$

The equations (11), (12) are called Lagrange equations, a simple way to obtain this equations is to form the equation

$$F = f + \sum_{r=1}^m \lambda_r \phi_r \quad (13)$$

And differentiate it with respect to $x_1, x_2, x_3, \dots, x_n$ keeping $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ as constants to get

$$F = \frac{\partial f}{\partial x_s} + \sum_{r=1}^m \lambda_r \frac{\partial \phi_r}{\partial x_s} = 0, \quad 1 \leq s \leq n, \quad 1 \leq r \leq m \quad (14)$$

Equation (14) gives the stationary values of $f(\underline{x})$, $\underline{x} = x_1, x_2, x_3, \dots, x_n$, subject to the constrains $\varphi_r(x_1, x_2, x_3, \dots, x_m) = 0$.

To determine the nature of this stationary values, define the following determinants

$$\Delta = \begin{vmatrix} F_{11} & \dots & F_{1n} & \phi_1^1 & \dots & \phi_1^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ F_{n1} & \dots & F_{nn} & \phi_1^m & \dots & \phi_1^m \\ \phi_1^1 & \dots & \phi_1^n & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ \phi_1^m & \dots & \phi_1^m & 0 & 0 & 0 \end{vmatrix}$$

And the minors $\Delta_1, \Delta_2, \Delta_3, \dots$ which obtained from Δ by eliminating the first row and column, second row and column and so on, then we have the following:

The sufficient conditions for $f(a_1, a_2, a_3, \dots, a_n)$ to be minimum are:

(i) If m is even , $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$, \dots , $\Delta_{n-m} > 0$

(ii) If m is odd , $\Delta_1 < 0$, $\Delta_2 < 0$, $\Delta_3 < 0$, \dots , $\Delta_{n-m} < 0$

And the sufficient conditions for $f (a_1 , a_2 , a_3 , \dots , a_n)$ to be maximum are:

(i) If m is even , $\Delta_1 > 0$, $\Delta_2 < 0$, $\Delta_3 > 0$, \dots , $(-1)^{n-m} \Delta_{n-m} < 0$

(ii) If m is odd , $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, \dots , $(-1)^{n-m} \Delta_{n-m} > 0$

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